Parallels in other geometries:

Spherical geometry:

Recall that a “line” on a sphere is a great circle—a circle produced by a Euclidean plane cutting the sphere in such a way that the plane passes through the center of the sphere.

1. If you think of the Earth as being a sphere, then the equator is a great circle—a spherical line. If you do the process of constructing a parallel line next to the equator on the sphere, what sort of spherical line do you get? Is it parallel to the equator? Why or why not?

2. Check out our proof that if alternate interior angles are congruent then the lines are parallel. Why does this fail on a sphere?

Hyperbolic geometry:

There are several models for hyperbolic geometry (a model is a way of showing what the geometry would look like. One of them is the surface approximated by the crocheted things we looked at in class. It’s easiest to understand because distances and angles are both what they appear to be to our Euclidean trained eyes.

Another model is Minkowski space which is used for expressing space-time in general relativity models.

The easiest model to draw on paper is the Poincare disk. For that , you just draw a disk and hyperbolic lines consist of diameters of the sphere and arcs of circles that meet the boundary circle of the disk at right angles. In the Poincare disk, angles are what they look like, but lengths are not. Things that look close together near the edge of the disk are really far apart. If you look at the Escher disk art, it will give you an idea of how distances work.

You can get a computer to do the Poincare disk work for you by using one of these programs:
http://www.geogebratube.org/material/show/id/7005
http://www.cs.unm.edu/~joel/NonEuclid/NonEuclid.html

3. Make a Poincare disk line l. Pick a point P near-ish the line, but not on it, and find a parallel m to l through P using our construction. Pick another point Q on m, and do the same construction to find a line parallel to l through Q. You should be getting a different line than you got last time. That means there are two parallels to l through point Q. In fact, each point-line pair has infinitely many parallels. (print your picture)

4. Read carefully Postulate 5 (the fifth geometry axiom—the one about parallels). Carefully draw a Poincare disk picture that shows that this postulate doesn’t hold in hyperbolic geometry (and therefore hyperbolic geometry doesn’t just look different from Euclidean geometry—it’s fundamentally a different geometry). (Your picture should satisfied the hypotheses—the “ifs” of the parallel postulate, but not the “thens”).\

A short history of geometry and axiom systems:

If you ask a mathematician what mathematics is (and you make it clear that you really want to know) they'll probably tell you that it consists of logically deducing results (about numbers or functions or shapes or whatever) from carefully stated assumptions. Logically building theorems (results) on axioms (carefully stated assumptions) is fundamental to what mathematics is today. In order for something to be "proven" in math, it has to be logically deduced from some stated assumptions. When done carefully, that means the results can be relied upon to always be true in any situations where the assumptions are true.

Way back when, geometry started out as results about shapes. The shapes fit together so nicely that mathematicians and philosophers started talking about why the results were true, and writing down ways of explaining why things were true. This had been going on for rather a long time (well over 100 years) when Euclid started writing (~300 BC). Euclid organized all of what he thought of as the most important and interesting geometric results in a way that everything built on things that had come before it. He started by explaining his assumptions—the things you needed to know or believe in order to understand his reasons. He was so good at organizing the results that for a very long time afterward (over 2000 years) other people writing about geometry would start with Euclid's works and explain how their results compared with Euclid. Lots of people wrote new things, and different things, and had ideas that they argued were better than how Euclid explained things, but the big structure they were comparing to was Euclid's Elements of Geometry.

One part of Euclid's assumptions that lots of mathematicians weren't quite happy with was the parallel postulate: it was just so long and complicated compared to the others, that it seemed as though it surely could be deduced from the other assumptions. And so people tried to show how the parallel postulate could be deduced from the other axioms, and always, often several years or even decades later, someone else would come along and show that they hadn't really done it. There was always some sneaky assumption that they had made in their argument that they hadn't noticed, and it turned out that that sneaky assumption was equivalent to the parallel postulate.

Two axioms equivalent if you can take an axiom system that is otherwise the same and substitute one axiom for the other and then the other axiom can be proven as a theorem. For example,

Euclid's parallel postulate says:

- If a line falling on two straight lines makes the interior angles on the same side less than two right angles, then the two lines intersect on that side of the line.

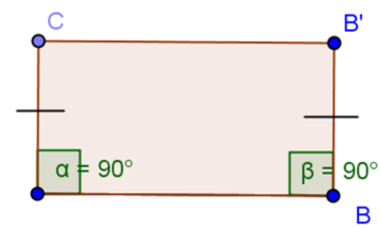
Playfair's axiom (which could also be called Proclus axiom) says:

- Given a line and a point not on the line, it is possible to draw exactly one line through the given point parallel to the line.

If you take the other axioms of Euclidean geometry plus Euclid's parallel postulate, you can prove Playfair's axiom as a theorem. If you take the other axioms of Euclidean geometry plus Playfair's axiom as an axiom, then you can prove Euclid's parallel postulate as a theorem. That means these axioms are equivalent—each proves the other.

So, along the way, while trying to prove that Euclid's parallel postulate wasn't needed, people proved that a lot of other statements were equivalent to that parallel postulate. As they did this, they got more and more comfortable working with if-then statements as axioms, and started to see if-then statements as a good way of stating axioms. They also got better at finding logical flaws and avoiding them, and all of these things reinforced a way of doing geometry (and mathematics) by logical deduction, and being very careful with stating assumptions (and looking for hidden assumptions).

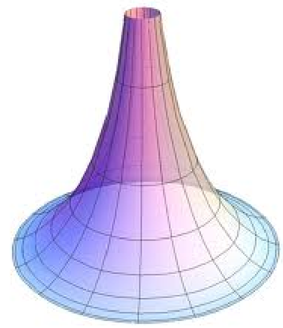
Some really cool new things happened in the time period starting just before 1700. Girolamo Saccheri tried to put together a proof by contradiction that the parallel postulate wasn't needed. He started with a quadrilateral that is now called a Saccheri quadrilateral: a quadrilateral on a base where the sides are congruent and perpendicular to the base, and then looking at the angles not on the base. Those angles are congruent. Saccheri proved that they couldn't both be obtuse (contradiction). He proved that if the angles were right angles, then he would have proved the parallel postulate. He tried very hard to prove that the angles weren't acute, but that didn't work out (another error for someone to catch). What it did do, though (because Saccheri was able to prove so many things) is that it got people interested in what would happen if the angles were acute. Could that happen? Would it keep on being consistent, and not give a contradiction?



So several other people went about proving more and more things that would be true if the angles were acute. Between 1810 and 1840 three separate mathematicians (Gauss, Lobachevsky and Bolyai) became convinced that there was no contradiction in the hypothesis that the angles in a Saccheri quadrilateral were acute—that replacing the parallel postulate with a version that allowed for acute angles in such a quadrilateral would yeild a separate and also consistent geometry (which is now most commonly called hyperbolic geometry). One big thing this meant was that the parallel postulate wasn't an extra. You really don't get the same sorts of shapes, results and space if you don't have the parallel postulate or you change the parallel postulate.

The most important thing this did for mathematics was that it convinced mathematicians that complicated if-then type axioms were necessary for carefully stating the assumptions of geometry and mathematics. Over the next two centuries, mathematicians started restating more and more mathematics in terms of axioms, and they were more careful to include all of the axioms that were needed. One of the most important things that happened (in Geometry) was David Hilbert's revised axiom system for Euclidean geometry (just before 1900)—it's an axiom system that has a similar structure to that of Euclid's original axioms, but its axioms have if-then type statements, and there are a lot more of them (since once people started looking for them, there were a lot of hidden assumptions in Euclid's organization that you could take and state overtly).

Another interesting thing that happened was that mathematicians wanted to prove that hyperbolic geometry was really consistent—that there were no hidden contradictions in this new geometry. It turns out to be really hard to prove that there are no contradictions in a set of axioms. We believe that Euclidean geometry doesn't have any hidden contradictions because we can draw dots and lines on paper and show how the Euclidean axioms all fit together in that physical system—so what we say is that we have model that we understand that shows the Euclidean axiom system, and because all of the axioms are true of the model, then it must be possible for all of the axioms to be simultaneously true. So what do do about hyperbolic geometry?



One of the first models for hyperbolic geometry was a surface in 3-space where the hyperbolic space axioms were true. There are a bunch more. Two nicely drawable ones are the upper half plane model (explained in the book) and the Poincare disk model. These two models have the advantage that everything in these models is described in terms of Euclidean geometry. That means you can prove the axioms of hyperbolic geometry in these models are consistent by using Euclidean geometry (so if Euclidean geometry is consistent, then hyperbolic geometry must also be consistent).

5. How did Euclid's Elements of Geometry lead to mathematicians using axioms to define mathematics?

6. State an alternate version of the parallel postulate (an axiom equivalent to the parallel postulate)

7. What is a **model** in mathematics