Geometry
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Some axioms and theorems (C.E. Burgess)

Axiom 1 Every line is a non-empty set of points.
Axiom 2. There exist at least two points
Axiom 3. If A and B are two points*, then there is a line which contains both A and B
*This sentence structure should be interpreted to imply that A and B are not the same point.
Axiom 4. If *l* is a line, then there is a point which does not belong to *l*.

**Question: what is the smallest example of a space that satisfies axioms 1-4?

Thm. 1. Every point belongs to at least two lines

Thm. 2. There exist at least 3 points

Thm. 3. There exist at least 3 lines

Thm 4. If there exists only 3 points, and each line contains at least one point, then there do not exist more than 6 lines.

Axiom 5. If A and B are two points, there do not exist two lines such that each of them contains both A and B.

Defn: Two sets *intersect* if they share at least one element. Two lines *intersect* if they share at least one point.

Thm. 5: If two lines intersect, then their intersection is a point.

Axiom 6. If L is a line, and P is a point that does not belong to L, then there do not exist two lines which contain P and do not intersect L

Defn. Two lines are <u>parallel</u> to each other if they do not intersect.

Thm. 6: If each of two lines L and M are parallel to a third line N, then L and M are parallel.

Thm. 7. If the lines L and M are parallel, and line N intersects L (and is different from L), then N intersects M.

Axiom 7. If L is a line and P is a point that does not belong to L, then there exists a line which contains P and is parallel to L.

Thm. 8. Every line contains at least two points

Thm. 9. No line is a proper subset of a line

Thm. 10. There exist at least 4 points.

Thm. 11. There exist 4 points such that no line contains 3 of them.

Thm. 12. Every point belongs to at least 3 lines.

Thm. 13. There exist at least 6 lines.

Examples to find:

a. A space with only 4 points
b. A space with only 6 lines
c. A space where no point belongs to more than 3 lines
d. A space where some one consists of only 2 points
e. A finite space with more than 4 points
f. A space that satisfies any 6 of the axioms, but not does not satisfy the remaining axiom.

Note: you may use algebra and the properties of real numbers.

Defn. Points are *collinear* of they are a subset of the same line.

Axiom 8: Given point A and point B*, there is a unique non-negative number $\rho(A,B)$ which is the *distance* from A to B. *A and B are allowed to be the same point.

Axiom 9: For any points A and B, $\rho(A,B)=\rho(B,A)$

Axiom 10: Given point A and point B $\rho(A,B)=0$ if and only if $A=B$.

Axiom 11: If points A, B, and C are non-collinear points, then $\rho(A,B)+\rho(B,C)>\rho(A,C)$

Example: Choose randomly an example from the set previously provided. Define $\rho(A,B)=0$ if $A=B$ and $\rho(A,B)=1$ if $A\neq B$. Does ρ satisfy axioms 8-11?

Defn, A point B is between two points A and C if it is different from both A and C and $\rho(A,B)+\rho(B,C)=\rho(A,C)$. This relationship will be denoted A-B-C.

Thm. 14. If a point B is between points A and C then A, B, and C are collinear.

Thm 15. If the point B is between two points A and C, then C is not between A and B.

Thm. 16. If the point B is between two points A and C, then B is also between C and A.

Axiom 12. If A and B are two points and r is a positive number, then there is a point C such that A-B-C and $\rho(B,C)=r$

Thm. 17. Every line contains at least 4 points

Thm. 18. If L is a line and A is a point of L, then there exist two points B and C of L such that B-A-C.

Thm 19. If L is a line and A is a point of L, then there exist two points B and C of L such that B-A-C.

Defn. A set consisting of four points A, B, C, and D is said to have the <u>order</u> A-B-C-D if A-B-C, B-C-D, A-C-D, A-B-D and $\rho(A,B)+\rho(B,C)+\rho(C,D)=\rho(A,D)$.

Thm. 20. A, B, C, D have order A-B-C-D if and only if D-C-B-A.

Thm 21. If four points have an order, then they are collinear.

Axiom 13. Every collinear set of 4 points has an order.

Thm 22. If A, B, and C are three collinear points, then one of them is between the other two.

Thm. 23. If A-B-C and B-C-D then A-B-D and A-C-D

Thm. 24. If A-B-C and A-C-D then A-B-D and B-C-D

Thm. 25. If A-B-D and A-C-D, then either B=C or A-B-C or A-C-B

Thm. 26. If A-B-C and A-B-D then either C=D or B-C-D or B-C-D.

Thm. 27. If A and B are two points, then there is a point between them.

Thm. 28. There do not exist 4 collinear points A, B, C, D, such that $\rho(A,B)=\rho(A,C)=\rho(A,D)$

Thm 29. If A, B, and C are points (not necessarily distinct), then $\rho(A,B)+\rho(B,C)\geq\rho(A,C)$

Thm 30. Every line contains infinitely many points

Thm 31. There is a 1-1 order preserving correspondence between the points of a line and the real numbers.

Defn. A set R of points is said to be a *ray* if there exist two points A and B such that a point X belongs to R if and only if one of the following is satisfied

a. X=A
b. X=B
c. A-X-B
d. A-B-X

The point A is called a *starting point* of R. This ray can be denoted $\overrightarrow{AB}$.

Thm. 32. No ray has two starting points.

Thm. 33. Every ray is a proper subset of a line.

Thm. 34. No ray is a line.

Thm. 35. If A is a point of a line L, then there exist two and only two rays which are subsets of L and have A as a starting point.

Thm. 36. If X and Y are two points of a ray R, then the starting point of R is not between X and Y.

Thm. 37. If C is a point of the ray $\overrightarrow{AB}$, and $C \neq A$ then $\overrightarrow{AC}=\overrightarrow{AB}$

Defn. If A and B are two points, then the set of all points between A and B is called an *open interval*, and the set consisting of A and B together with all points between A and B is called a *closed interval* and a segment and can be written $\overline{AB}$. A and B are the *endpoints* of the interval or segment

Thm. 38 No open interval is a closed interval.

Thm. 39. Every interval is a proper subset of a line

Thm. 40. No interval is a line.

Thm. 41. No interval is a ray.

Note: Thm. 38 is not true for spaces which satisfy only axioms 1-11.

Thm. 42. No interval has more than two endpoints.

Defn. A point C is a *midpoint* of an interval if $\rho(A,C) = \rho(C,B)$ and A-C-B.

Thm. 43. Every interval has one and only one midpoint.

Defn. A set of points is said to be an angle if H is the union of two rays which have the same starting point; that is, there exist two rays $\overrightarrow{BA}$ and $\overrightarrow{BC}$ such that a point X belongs to H if and only if X belongs to at least one of the rays $\overrightarrow{BA}$ and $\overrightarrow{BC}$. This angle will be denoted $\measuredangle ABC$. The point B is called a *vertex* of $\measuredangle ABC$ if A, B, are non-collinear. $\measuredangle ABC$ is called a *straight* angle if A, B, and C are collinear.

Thm. 44. Every straight angle is a line.

Thm. 45. No angle has two vertices.

Thm. 46. If, in $\measuredangle ABC$, D is a point different from B on $\overrightarrow{BA}$ and E is point different from B on $\overrightarrow{BC}$, then $\measuredangle DBE = \measuredangle ABC$.

Defn. A set H of points is said to be a *triangle* if there exist three non-collinear points A, B, C such that H is the union of the three segments $\overline{AB}$, $\overline{BC}$, $\overline{AC}$; that is, a point X belongs to H if and only if X belongs to at least one of the segments $\overline{AB}$, $\overline{BC}$, $\overline{AC}$. Such a triangle will be denoted $\triangle ABC$. Each of the open intervals AB, AC, BC is called a *side* of $\triangle ABC$, and each of the points A, B, C is called a *vertex* of $\triangle ABC$.

Thm. 47. If a line L intersects two sides of $\triangle ABC$ then none of A, B, C belongs to L.

Thm. 48. If the line L intersects the open interval (side) AB of $\triangle ABC$, and L contains C, then L does not contain A or B.

Thm. 49. If a line L contains two vertices A, B of $\triangle ABC$, then L does not intersect either of the open intervals AC or BC.

Thm. 50. Every triangle has 3 and only 3 vertices.

Thm. 51 Every triangle has 3 and only 3 sides.

Axiom 14. (Pasch) If A, B, C are three non-collinear points and L is a line which contains a point between A and B and does not contain any of the points A, B, and C, then either L contains a point between A and C or L contains a point between B and C.

Thm 52. if a line L intersects a side of a triangle $\triangle ABC$, then L contains at least two points of $\triangle ABC$.

Note that Theorem 52 is equivalent to Axiom 14.

Thm. 53. No line intersects all 3 sides of a triangle.

Thm. 54. if L is a line, then there exist two sets S_1 and S_2 such that

i. every point belongs to one of the sets S_1, S_2 or L
ii. no point belongs to two of the sets S_1, S_2 or L
iii. no point of L is between two points of S_1
iv. no point of L is between two points of S_2
v. for each point X_1 in S_1 and each point X_2 in S_2, there is a point of L between X_1 and X_2.

Defn. A set S is said to be a *side* of a line L if there exist two sets S_1 and S_2 such that they and L satisfy the five requirements in the conclusion of theorem 54 and S is one of the two sets S_1, S_2.

Thm. 55. No line has three sides.

Thm 56. If A, B, C are three non-collinear points and D and E are points such that A-B-D and B-E-C, then there exists a point F such that D-E-F and A-F-C.

Thm. 60. If A, B, C are three non-collinear points and D and F are points such that A-B-D and A-F-C, then there exists a point E such that B-E-C and D-E-F.

Thm. 61. If a line L contains the starting point A of a ray R and R is not a subset of L, then there is a side of L which contains every point of F different from A.

Thm. 62. If L_1 and L_2 are two lines having a common point B, then there exist two points A and C on L_1 such that A and C belong to different sides of L_2.

Thm. 63. If L is a line and A is a point which does not belong to L, then the side of L which contains A is called the *A-side* of L.

Defn. A set I is said to be the *interior* of $\measuredangle ABC$ if

i. A, B, C are non-collinear and
ii. I consists of all points that belong to both the C-side of the line that contains A and B, and to the A-side of the line that contains B and C.

Defn. The set E is said to be the *exterior* of $\measuredangle ABC$ if

i. A, B, C are non-collinear and
ii. E consists of all points that belong neither to $\measuredangle ABC$ nor to the interior of $\measuredangle ABC$

Thm. 64. If X and Y are two points which belong to the interior of $\measuredangle ABC$, then every point of the segment $\overline{XY}$ belongs to the interior of $\measuredangle ABC$

Thm. 65. If X and Y are two points of the exterior of $\measuredangle ABC$, then there are at most two points of $\measuredangle ABC$ between X and Y.

Thm. 66. If X is a point of the interior of $\measuredangle ABC$, then there is one and only one point of $\measuredangle ABC$ between X and Y.

Thm. 67. If P is a point of the interior of $\measuredangle ABC$, then there exists a point D on $\overrightarrow{BA}$ and a point E on $\overrightarrow{BC}$ such that D-P-E.

Thm. 68. If A, B, C are three non-collinear points, and D is a point different from B on $\overrightarrow{BA}$ and E is a point different from B on $\overrightarrow{BC}$, then every point between D and E belongs to the interior of $\measuredangle ABC$.

Defn. The *interior* of $\triangle ABC$ consists of all points which are interior to all three of the angles $\measuredangle ABC$, $\measuredangle ACB$ and $\measuredangle BAC$. The *exterior* of the triangle consists of all points which belong neither to the triangle, nor to its interior.

Thm. 69. If a point belongs to the interior of $\measuredangle ABC$ and to the interior of $\measuredangle ACB$, then it belongs to the interior of $\triangle ABC$.

Thm. 70. If X and Y are two points of $\triangle ABC$, then there is no point of $\triangle ABC$ between X and Y

Thm. 71. If X and Y are two points of the exterior of $\triangle ABC$, then either there are no more than two points of $\triangle ABC$ between X and Y or there is a side of $\triangle ABC$ between X and Y

Thm. 72. If X and Y are two points of the exterior of triangle $\triangle ABC$, then there exists a point W of the exterior such that $\overline{XW}$ and $\overline{YW}$ are subsets of the exterior.

Thm. 73. If X and Y are points of the interior and exterior, respectively, of a triangle $\triangle ABC$, then there is one and only one point of $\triangle ABC$ between X and Y.

Thm. 74, If a ray R intersects the interior of a triangle $\triangle ABC$, then R intersects $\triangle ABC$.

Thm. 75. If X is a point of the exterior of triangle $\triangle ABC$, then there is a line that contains X and lies entirely in the exterior of $\triangle ABC$.

Thm. 76. If A is a point of a side S if a line then there exists a line which contains A and is a subset of S.

Thm. 77. If A is a point of the interior I of an angle, then there exists a ray which contains A and is a subset of I.

Thm. 78. If A is a point of the exterior of an angle, then there exists a line which contains A and lies entirely in the exterior.

Thm. 79. If a line intersects the interior of an angle, then it intersects the angle.

Defn. An angle $\measuredangle ABC$ is *congruent* to angle $\measuredangle A'B'C'$ if for every pair of points D and E on rays $\overrightarrow{BA}$ and $\overrightarrow{BC}$ respectively, and for any two points D' and E' on the rays $\overrightarrow{B'A'}$ and $\overrightarrow{B'C'}$ respectively such that $\rho(B,D)=\rho(B',D')$ and $\rho(B,E)=\rho(B',E')$, it follows that $\rho(D,E)=\rho(D',E')$. Congruent angles will be denoted: $\measuredangle ABC \cong \measuredangle A'B'C'$.

Thm. 80. Every angle is congruent to itself.

Thm. 81. Every two straight angles are congruent.

Thm. 82. Every angle that is congruent to a straight angle is a straight angle.

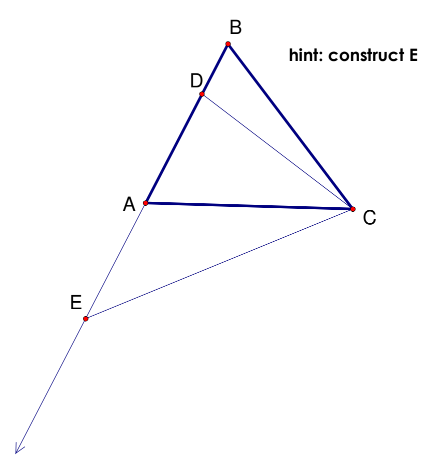
Thm. 83. If $\measuredangle ABC \cong \measuredangle A'B'C'$ and $\measuredangle A'B'C' \cong \measuredangle A''B''C''$, then $\measuredangle ABC \cong \measuredangle A''B''C''$

Axiom 15. If

i. A, B, C are three non-collinear points
ii. A',B',C' are three non-collinear points
iii. D is a point such that A-B-D
iv. D' is a points such that $A'-B'-D'$
v. $\rho(A,B)=\rho(A',B')$
vi. $\rho(B,C)=\rho(B',C')$
vii. $\rho(A,C)=\rho(A',C')$
viii. $\rho(B,D)=\rho(B',D')$

then $\rho(C,D)=\rho(C',D')$

Thm. 85.



ix. A, B, C are three non-collinear points
x. A',B',C' are three non-collinear points
xi. D is a point such that A-D-B
xii. D' is a points such that $A'-D'-B'$
xiii. $\rho(A,B)=\rho(A',B')$
xiv. $\rho(B,C)=\rho(B',C')$
xv. $\rho(A,C)=\rho(A',C')$
xvi. $\rho(B,D)=\rho(B',D')$

then $\rho(C,D)=\rho(C',D')$

Thm. 86. If, for $\measuredangle ABC$ and $\measuredangle A'B'C'$, there exist two points D and E different from B on the rays $\overrightarrow{BA}$ and $\overrightarrow{BC}$ respectively and D' and E' on rays $\overrightarrow{B'A'}$ and $\overrightarrow{B'C'}$ respectively, such that $\rho(B,D)=\rho(B',D')$, $\rho(B,E)=\rho(B',E')$ and $\rho(E,D)=\rho(E',D')$, then $\measuredangle ABC \cong \measuredangle A'B'C'$

Defn. A triangle *H* is said to be <u>congruent</u> to a triangle H' if the vertices of *H* can be labelled with A, B, C and the vertices of H' with A', B', C' such that

i. $\rho(A,B) = \rho(A',B')$
ii. $\rho(C,B) = \rho(C',B')$
iii. $\rho(A,C) = \rho(A',C')$
iv. $\measuredangle ABC \cong \measuredangle A'B'C'$
v. $\measuredangle BAC \cong \measuredangle B'A'C'$
vi. $\measuredangle ACB \cong \measuredangle A'C'B'$

We denote this $\triangle ABC \cong \triangle A'B'C'$

Thm 87 (SSS) If for $\triangle ABC$ and $\triangle A'B'C'$, $\rho(A,B) = \rho(A',B')$, $\rho(C,B) = \rho(C',B')$, $\rho(A,C) = \rho(A',C')$, then $\triangle ABC \cong \triangle A'B'C'$.

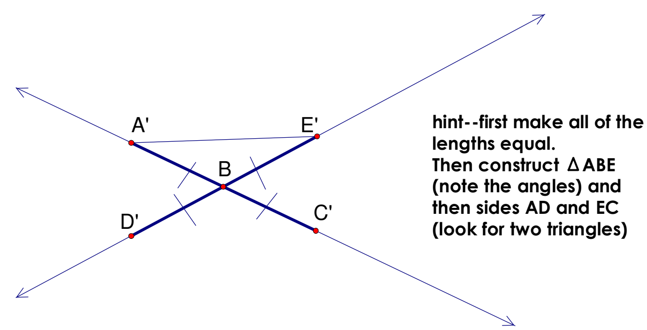
Thm. 88 (SAS) If for $\triangle ABC$ and $\triangle A'B'C'$, $\rho(A,B) = \rho(A',B')$, $\rho(C,B) = \rho(C',B')$, $\measuredangle ABC \cong \measuredangle A'B'C'$, then $\triangle ABC \cong \triangle A'B'C'$.

Thm, 89 If for $\triangle ABC$, $\rho(A,B) = \rho(A,C)$, then $\measuredangle ABC \cong \measuredangle ACB$.

Thm. 90 If for $\triangle ABC$, $\measuredangle ABC \cong \measuredangle ACB$, then $\rho(A,B) = \rho(A,C)$

Defn. A triangle for which $\rho(A,B) = \rho(A,C)$ is an <u>isosceles</u> triangle. Sides $\overline{AB}$ and $\overline{AC}$ are the <u>legs</u> of the isosceles triangle, and side $\overline{BC}$ is the <u>base</u>. The angles $\measuredangle ABC$ and $\measuredangle ACB$ are the <u>base angles</u> and $\measuredangle BAC$ is the <u>vertex angle</u>.

Thm. 91: If ℓ_1 and ℓ_2 are two lines, *B* is the point of intersection of ℓ_1 and ℓ_2, *A* and *C* are points of ℓ_1 such that *A*-*B*-*C*, and *D* and *E* are points of ℓ_2 such that *D*-*B*-*E*, then $\measuredangle ABD \cong \measuredangle CBE$



Defn. If A, B, and C are points of a line L such that A-B-D and D is a point not on L, then $\measuredangle ABC$ and $\measuredangle CBD$ are called <u>supplementary angles</u> and each of them is called a <u>supplement</u> of the other.

Thm. 92. Two angles are congruent if they have congruent supplements.

Axiom 16/Thm. 93: If A' and B' are points of a line L, and S is a side of L, and A, B, C are non-collinear points such that $\rho(A,B) = \rho(A',B')$, then there exists a point C' of S such that $\rho(A'C') = \rho(A,C)$ and $\rho(C,B) = \rho(C',B')$

Thm. 94. If $\measuredangle ABC$ is not a straight angle, A' and B' are points of a line L, and S is a side of L,, then there exists a point $C' \in S$ such that $\measuredangle ABC \cong \measuredangle A'B'C'$

Axiom 17/Thm 95: If A' and B' are points of a line ℓ, and A, B, C are non-collinear points, and C' and C'' are two points such that

- $\rho(A', B') = \rho(A, B)$
- $\rho(A', C') = \rho(A', C'') = \rho(A, C)$
- $\rho(B', C') = \rho(B', C'') = \rho(B, C)$

then C' and C'' belong to different sides of ℓ.

Theorem 96. If A, B, C are non-collinear points, D is on the C-side of the line $\overleftrightarrow{AB}$, and $\measuredangle ABC = \measuredangle ABD$, then D is on the ray $\overrightarrow{BC}$.

Thm. 97 (ASA). Given triangles $\triangle ABC$ and $\triangle A'B'C'$ such that $\rho(A, B) = \rho(A'B')$, $\measuredangle BAC \cong \measuredangle B'A'C'$ and $\measuredangle ABC = \measuredangle A'B'C'$ then $\triangle ABC \cong \triangle A'B'C'$

Thm. 98. If the point D belongs to the interior of $\measuredangle ABC$, then $\measuredangle ABC$ is not congruent to $\measuredangle ABD$.

Defn. An angle is a *right angle* if it is congruent to one of its supplements.

Thm. 99. There exists a right angle

Thm. 100. Every angle that is congruent to a right angle is a right angle.

Thm. 101. Every two right angles are congruent.

Defn. Two lines *l* and *m* are *perpendicular* to each other if there exist three points *A, B, C* such that $A, B \in l$, $B, C \in m$ and $\measuredangle ABC$ is a right angle.

Thm. 102. If the two lines *l* and *m* are perpendicular to a line *n*, then *l* and *m* are parallel.

Thm. 103. If two lines *l* and *m* are parallel, and line *n* is perpendicular to *l*, then *n* is also perpendicular to *m*.

Thm. 104. Given lines *l* and *m* such that $A, B, C \in l$, $D, E, F \in m$, $A - B - C$, $D - E - F$, $\measuredangle ABE \cong \measuredangle BEF$, and *A, D* on the same side of $\overleftrightarrow{BE}$ and *C, F* on the same side of $\overleftrightarrow{BE}$ (alternate interior angles are congruent), then *l* is parallel to *m*.

Thm. 105. Given parallel lines *l* and *m* that are intersected by line *n*, the alternate interior angles are congruent.

Thm 106. Given supplementary angles $\measuredangle ABC$ and $\measuredangle CBD$ and angles congruent to those angles $\measuredangle ABC \cong \measuredangle EFG$ and $\measuredangle CBD \cong \measuredangle GFH$ such that H and E are on opposite sides of $\overleftrightarrow{GF}$ then $\measuredangle EFG$ and $\measuredangle GFH$ are supplements.

Thm. 107. Given angles that share a side $\measuredangle ABC$ and $\measuredangle CBD$ such that C lies in the interior of $\measuredangle ABD$, and angles congruent to those angles $\measuredangle ABC \cong \measuredangle EFG$ and $\measuredangle CBD \cong \measuredangle GFH$ such that G lies in the interior of $\measuredangle EFH$, then $\measuredangle ABD \cong \measuredangle EFH$.

Defn. A set H of points is said to be a *quadrilateral* if there exist four points, A, B, C, D such that no three are collinear, and the segments $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{AD}$ intersect only at the endpoints, and H is the union of the four segments $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{AD}$. A quadrilateral has four sides $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{AD}$, and four angles $\measuredangle ABC$, $\measuredangle BCD$, $\measuredangle CDA$ and $\measuredangle DAB$. Sides of a quadrilateral are *adjacent* if they share an endpoint, and *opposite* if they do not share an endpoint. Angles of a quadrilateral are *adjacent* if they share a side (have as a subset one of the sides of the quadrilateral), and *opposite* if they do not share a side.

Defn. A square is a quadrilateral whose angles are all right angles, and whose sides are all congruent (have the same length)

Defn. A rectangle is a quadrilateral whose angles are all right angles.

Defn. A parallelogram is an a quadrilateral whose opposite sides are parallel (both pairs of opposite sides)

Defn. A rhombus is a quadrilateral whose sides are all congruent

Defn. A kite is a quadrilateral with two (disjoint) pairs of congruent sides.

Thm. 108. A quadrilateral is a parallelogram if and only if its opposite sides are congruent.

Thm. 109. A quadrilateral is a parallelogram if and only if all adjacent pairs of angles of the quadrilateral are congruent to pairs of supplementary angles.

Thm. 110. A quadrilateral is a parallelogram if and only if it's diagonals bisect each other.

Axiom 18/Thm. 111: Every angle $\measuredangle ABC$ has a measure $m\measuredangle ABC$ which is a real number in the interval $(0,180]$ (we will denote angle measurements with the degree symbol º) with the properties that

- an angle has measure 180º if and only if it is a straight angle
- angles are congruent if and only if their measure is the same
- the sum of the measures of supplementary angles is 180º
- given an angle $\measuredangle ABC$ that is not a straight angle, and given D in the interior of $\measuredangle ABC$ then $m\measuredangle ABD + m\measuredangle DBC = m\measuredangle ABC$

Thm. 112. The sum of the measures of the angles in a triangle is the measure of a straight angle.

Thm. 113. A quadrilateral is a parallelogram if and only if the opposite angles are congruent.

Defn. A segment $\overline{AB}$ is bisected by a set S (line, segment, ray, etc.) if the intersection of the segment and the set is the midpoint of the segment.

Defn. An angle $\measuredangle ABC$ is bisected by a straight object (segment, ray, line) if the straight object includes the vertex B of the angle, and a point D of the interior of the angle such that $\measuredangle ABD \cong \measuredangle DBC$

Thm. 114 If one angle formed by two intersecting lines is a right angle, then all four angles formed by two intersecting lines are right angles.

Defn. Two lines which meet at right angles are called *perpendicular* lines.

Thm. 115 A quadrilateral is a rhombus if and only if its diagonals bisect each other and are perpendicular.

Thm. 116. A quadrilateral is a rhombus if and only if its diagonals bisect the angles of the quadrilateral.

Thm. 117. If a quadrilateral has congruent angles, the angles are right angles.

Thm. 118. A quadrilateral is a square if and only if its diagonals are congruent, bisect each other, and are perpendicular.

Which of the axioms are satisfied by each of these spaces?
Ex. 1: Points: A, B, C
Lines {A}, {B}, {C}, {A,B}, {A,C}, {B,C}

Ex. 2: Points: A, B, C
Lines {A,B}, {A,C}, {B,C}

Ex. 3: Points: A, B, C
Lines {A}, {B}, {C}

Ex. 4: Points: A, B, C, D
Lines {A}, {B}, {C}, {D}, {A,B}, {A,C}, {B,C}, {AD}, {B,D}, {C,D}, {A,B,C}, {A, B, D}, {A, C, D}, {B, C, D}, {A, B, C, D}

Ex. 5: Points: A, B, C, D
Lines {A}, {B}, {C}, {D}, {A,B}, {A,C}, {B,C}, {AD}, {B,D}, {C,D}, {A,B,C}, {A, B, D}, {A, C, D}, {B, C, D}

Ex. 6: Points: A, B, C, D
Lines {A}, {B}, {C}, {D}, {A,B}, {A,C}, {B,C}, {AD}, {B,D}, {C,D}, {A,B,C}, {A, B, D}, {A, C, D}, {B, C, D}

Ex. 7: Points: A, B, C, D
Lines {A,B}, {A,C}, {B,C}, {AD}, {B,D}, {C,D}

Ex. 8: Points: A, B, C, D
Lines: {A,B,C}, {A, B, D}, {A, C, D}, {B, C, D}

Ex. 9: Points: all of the points on the (surface of a) sphere
Lines: all of the great circles (intersection of the sphere with a Euclidean plane including the center of the sphere)

Ex. 10: Points: all of the points in $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$
Lines: all of the open segments that consist of the intersection of a Euclidean line with the point set.

Ex. 11: The Euclidean plane, with the usual points and lines

Ex. 12: Points: A, B, C, D
Lines: {A,B}, {A,B,C}, {A,B,D},{C,B,D}

Mad Libs version of the axioms:

Axiom 1 Every line is a non-empty set of points.

Axiom 2. There exist at least two points

Axiom 3. If $\underset{\text{point 1}}{\_\_\_\_\_}$ and $\underset{\text{point 2}}{\_\_\_\_\_}$ are two points*, then there is a line which contains** both $\underset{\text{point 1}}{\_\_\_\_\_}$ and $\underset{\text{point 2}}{\_\_\_\_\_}$

*This sentence structure should be interpreted to imply that A and B are not the same point.
**contains does not mean consists of

Axiom 4. If $\underset{\text{line}}{\_\_\_\_\_}$ is a line, then there is a point which does not belong to $\underset{\text{line}}{\_\_\_\_\_}$.

Axiom 5. If $\underset{\text{point 1}}{\_\_\_\_\_}$ and $\underset{\text{point 2}}{\_\_\_\_\_}$ are two points, there do not exist two lines such that each of them contains both $\underset{\text{point 1}}{\_\_\_\_\_}$ and $\underset{\text{point 2}}{\_\_\_\_\_}$.

Axiom 6. If $\underset{\text{line}}{\_\_\_\_\_}$ is a line, and $\underset{\text{point}}{\_\_\_\_\_}$ is a point that does not belong to $\underset{\text{line}}{\_\_\_\_\_}$, then there do not exist two lines which contain $\underset{\text{point}}{\_\_\_\_\_}$ and do not intersect $\underset{\text{line}}{\_\_\_\_\_}$.

Axiom 7. If $\underset{\text{line}}{\_\_\_\_\_}$ is a line and $\underset{\text{point}}{\_\_\_\_\_}$ is a point that does not belong to $\underset{\text{line}}{\_\_\_\_\_}$, then there exists a line which contains $\underset{\text{point}}{\_\_\_\_\_}$ and is parallel to $\underset{\text{line}}{\_\_\_\_\_}$.

Geometries:

1. Consider the example of spherical geometry:

Ex. 9: Points: all of the points on the (surface of a) sphere

Lines: all of the great circles (intersection of the sphere with a Euclidean plane including the center of the sphere)

A. Which axioms does it satisfy?

B. Suggest alternate axioms to replace the axioms it does not satisfy.

2. Consider Ex. 10:

Points: all of the points in $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$

Lines: all of the open segments that consist of the intersection of a Euclidean line with the point set.

A. Which axioms does it satisfy?

B. Suggest alternate axioms to replace the axioms it does not satisfy.

3. Consider the example of hyperbolic geometry as described in the textbook section 1.2

Points: all of the points in the open upper half-plane: $\{(x, y) \in \mathbb{R}^2 \mid y > 0\}$

Lines come in two varieties: vertical lines $LV_a = \{(a, y) \in \mathbb{R}^2 \mid y > 0\}$ and circles whose center lies on the x-axis: $LC_a = \{(x, y) \in \mathbb{R}^2 \mid (x-a)^2 + y^2 = r^2, y > 0\}$

A. Which axioms does it satisfy?

B. Suggest alternate axioms to replace the axioms it does not satisfy.