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An axiom system based on transformations for Euclidean Geometry

Points are objects in a space called the ***plane*** which can be written E^2. A ***line*** is a set of points. Points and lines are otherwise undefined, except that they must satisfy the axioms below:

Line axiom: Given any two points, there is one and only one straight line that contains both points.

Between-ness axiom: If *A, B, C* are three points in a line then exactly one of them is ***between*** the other two.

Distance Axiom: There is a distance function $d: E^2 \times E^2 \to \mathbb{R}$ (where E^2 is the plane), such that

i. $d(P,P)=0$ for every point *P* on the plane.

ii. $d(A,B)>0$ if *A* and *B* are distinct points. distinct means not the same points

iii. For three distinct points *A, B, C,* $d(A,B)+d(B,C)=d(A,C)$ if and only if *B* lies between *A* and *C* on the line $\overleftrightarrow{AC}$.

iv. For three distinct points *A, B, C,* if *B* does not lie between *A* and *C* or if *B* does not line on $\overleftrightarrow{AC}$, then $d(A,B)+d(B,C)>d(A,C)$

A ***circle*** with center $P \in E^2$ and radius $r \in \mathbb{R}$ is the set of all points $X \in E^2$ such that $d(X,P)=\mathbb{R}$

Separation Axiom: The infinite straight line, the triangle the circle and a pair of distinct rays that share an endpoint, ***separate*** the plane into two regions such that any line or arc joining a point in one region to a point in the other region intersects the separating figure. These regions are called ***sides***.

intersects means shares at least one point

[The definitions of segment and ray are omitted from this list, but are the usual definitions]

An ***angle***, consists of two rays with a common endpoint, and a side of the plane separated by the rays.

Angle Axiom: Given an angle $\angle ABC$, there is a function *m* called the ***measure of the angle*** that maps angles in the plane to real numbered degrees between 0° and 360° inclusive, with the properties that :

i. The trivial angle, consisting of a single ray and itself has measure 0° if the associated region is empty and 360° if the associated region together with the ray comprise the whole plane. Non-trivial angles have measures strictly between 0° and 360°.

ii. Given two angles who share a ray and whose regions do not intersect, the sum of the measures of the angles is the measure of the angle whose sides are the non-shared sides of the angles, and whose region consists of the regions of the two angles and the shared ray.

iii. Given two rays that comprise a line, the measure of the angle is 180°.

Definition: An ***isometry*** is a 1-1, onto function that maps the plane to itself, such that distances and angle measures are preserved. The image of a region under an isometry is called its ***isometric image.*** (If $f(A)=A'$, $f(B)=B'$ and $f(C)=C'$ then distances are preserved if $m(\overline{AB})=m(\overline{A'B'})$, and angle measures are preserved if $m\angle BAC = m\angle B'A'C'$). A point *P* is ***fixed*** by a function *f* if $f(P)=P$.

Isometries Axiom:

i. Given a point (*P*) and an angle ($\angle ABC$), there is a unique isometry (*f*) called a ***rotation*** that fixes the given point and then angle formed by any point, the fixed point, and the image of the point is congruent to the given angle ($\angle XPf(X) = \angle ABC$).

ii. Given a line, there is a unique isometry called a ***reflection*** that fixes points on the line and maps points on one side of the line to the other side of the line.

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Definition: Two **segments** are ***congruent*** if they have the same length. Two **angles** are ***congruent*** if they have the same angle measure.

Definition: Two point sets in the plane are ***congruent*** if there is an isometry that maps one to the other. (Named objects are congruent if they are congruent as point sets and if their named parts correspond, eg. $\Delta ABC \cong \Delta DEF$ means that there is an isometry, $f : \Delta ABC \to \Delta DEF$ such that $f(A) = D, f(B) = E, f(C) = F$

Theorem 1 (Sides of Congruent Angles): If two congruent angles share a side (ray) and the other sides (rays) lie on the same side of the shared side, then their other sides are also shared. **I.e.** If $\angle ABC \cong \angle ABD$ and *C* and *D* lie on the same side of $\overleftrightarrow{AB}$ then $\overrightarrow{BC} = \overrightarrow{BD}$.

Theorem 2 (Endpoints of Congruent Segments): On a ray, there is at most one point at a given distance from the endpoint of the ray. **I.e.** If $C \in \overrightarrow{AB}$ and $\overline{CA} \cong \overline{BA}$ then $C = B$.

Theorem 3 (Line intersection): Any two distinct lines intersect in at most one point.

Theorem 4: If *f* is an isometry and A, B, C are collinear (on the same line), then $f(A), f(B), f(C)$ are collinear

Theorem 5: If *f* is an isometry, and *X* is on the circle with center *P* and radius *r*, then *f(X)* is on the circle with center *f(P)* and radius *r*.

A ***function*** $f : A \to B$ is a (often infinite) set of ordered pairs $\{(x, f(x)) \mid x \in A, f(x) \in B\}$, such that for any element $a \in A$, there is a unique ordered pair $(a, f(a))$, where *a* is the first coordinate of the ordered pair. Informally, we say that *f* is a rule that assigns to each element of *A* a unique element of *B*. The uniqueness property can be proved by proving that, for any pair of ordered pairs $(a, f(a)), (a',(f(a'))$, if $a = a'$ then $f(a) = f(a')$.

A function $f : A \to B$ is ***onto*** if for every $b \in B$ there is an element $a \in A$ such that *f(a)=b*.

A function $f : A \to B$ is ***1-to-1*** if for any $a, a' \in A$, if $f(a) = f(a')$ then $a = a'$. The informal notion of this is the contrapositive of this definition: if two elements of A are different, then their images are different.

Theorem 6: Let $f : A \to B$ and $g : B \to C$ be 1-1 functions, then $g(f) : A \to C$ is a 1-1 function.

Theorem 7: Let $f : A \to B$ and $g : B \to C$ be onto functions, then $g(f) : A \to C$ is an onto function.

Theorem 8: Let $f : A \to B$ and $g : B \to C$ be isometries, then $g(f) : A \to C$ is an isometry

Theorem 9 (Application): Given triangles ΔABC and ΔDEF there is an isometry that maps A to D, and maps B to a point on $\overrightarrow{DE}$, and maps C to a point to the side of $\overleftrightarrow{DE}$ that contains point F.

Theorem 10 (SAS): If two sides of one triangle are congruent to two corresponding sides of another triangle, and the angles contained by those sides are congruent, then the triangles are congruent.

If: ΔABC and ΔDEF such that $\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$ and $\angle ABC \cong \angle DEF$

Then: $\Delta ABC \cong \Delta DEF$

Theorem 11 (ASA): If two angles of one triangle and the segment between them are congruent to the corresponding angles and side of another triangle then the triangles are congruent.

Given:

Then:

Theorem 12: (segments can be duplicated): Given a ray starting at a given point, and a distance between two points, there exists another point that lies on the given ray such that the segment between it and the starting point of the ray is equal to the given distance.

Given:

Then:

Theorem 13 (angles can be duplicated): Given a non-trivial angle, a ray, and a side of the line containing the ray, there exists another ray such that the rays together are congruent to the given angle.

Given:

Then:

Theorem 13.5 (CPCTC): Corresponding sides and angles of congruent triangles are congruent

A triangle is **Isosceles** if (at least) two sides are congruent. In an isosceles triangle, the **base** refers to the side which is not identified as one of the two congruent sides, the base angles are the angles which share a side with the base, and the vertex angle is the angle between the two congruent sides.

Theorem 14 (equal sides implies equal angles): In an isosceles triangle, the base angles are congruent.

Given:

Then:

Theorem 15 (equal angles implies equal sides): In a triangle, if two angles are congruent, then the sides opposite those angles are also congruent.

Given:

Then:

Theorem 16 (SSS): If all three sides of a triangle are congruent to the corresponding sides of another triangle, then the triangles are congruent.

Given:

To prove:

Definition: A point M is a ***midpoint*** of a segment $\overline{AB}$ if it lies on the segment $\overline{AB}$ and its distances from the two endpoints are equal: $\overline{AM} \cong \overline{BM}$

Definition: Two intersecting lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$ are perpendicular if the angle formed by the lines is half of a straight angle ($180°$). Two segments are perpendicular if their extended lines are perpendicular.

Definition: A ***perpendicular bisector*** of a segment is a line, segment or ray that contains the midpoint of the segment, and is perpendicular to the segment.

Theorem 17: For any segment, there exists one and only one midpoint and one and only one perpendicular bisector.

Part 1: For any segment, these exists a midpoint and a perpendicular bisector.

Given:

To prove:

Part 2: For any segment, there can't be more than one midpoint:

Given:

To prove:

Part 2: For any segment, there can't be more than one perpendicular bisector:

Given:

To prove:

Theorem 18: A point lies on the perpendicular bisector of a segment if and only if it is equidistant from the endpoints of the segment.

Given:

To prove:

Theorem 19: Given a line and a point, there is one and only one line that goes through the point and is perpendicular to the line.

Given:

To prove:

Definition: A ***bisector*** of an angle $\angle ABC$ is a line, segment or ray that contains the vertex B of the angle, and a point D in the interior of the angle, such that $\angle ABD \cong \angle DBC$

Theorem 20: For any angle, there exists one and only one angle bisector. (prove for the case where the angle is less than 90°)

Given:

To prove: