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An axiom system based on transformations for Euclidean Geometry

Axiom 1 (lines): Given any two points, there is one and only one straight line that contains both points.
Note: lines are sets of points

Axiom 2 (Separation and between-ness): The infinite straight line, the triangle and the circle ***separate*** the plane into two regions such that any line or arc of a circle joining a point in one region to a point in the other region intersects the separating figure. These regions are called ***sides***.

A point on an infinite straight line divides the line into two regions, where the point lies ***between*** the regions and between pairs of points (one in each region). Given any three points on a line, exactly one of the points will lie between the other two.

Definition: A point on a line, together with one of the sides of the point on the line is a ***ray***. A ***segment*** between two points called endpoints consists of the two endpoints and all of the points on the line containing the endpoints that lie between the two points.

Axiom 3 (Existence of lengths and angles):

i. **Triangle Inequality** Given any two points A and B, there is a non-negative real number that is called the ***distance*** between them, with the properties that:
   - *a.* Any point is distance 0 from itself, and is a positive distance from any other point.
   - *b.* If C is a point on a segment, then the sum of the distances from the endpoints to C is equal to the distance between the endpoints.
   - *c.* If C is a point not on a given segment, then the sum of the distances from the endpoints to C is greater than the distance between the endpoints.

ii. Given an ***angle***, consisting of two rays with a common endpoint, and a choice of the regions separated by the two rays, then there is a real number between 0° and 360° that is called the ***measure of the angle***, with the properties that :
   - *a.* The trivial angle, consisting of a single ray and itself has measure 0° if the associated region is empty and 360° if the associated region together with the ray comprise the whole plane. Non-trivial angles have measures strictly between 0° and 360°.
   - *b.* Given two angles who share a ray and whose regions do not intersect, the sum of the measures of the angles is the measure of the angle whose sides are the non-shared sides of the angles, and whose region consists of the regions of the two angles and the shared ray.
   - *c.* Given two rays that comprise a line, the measure of the angle is 180°.

*Definition***:** An ***isometry*** (also called a ***rigid motion***) is a 1-1, onto function that maps the plane to itself in such a way that distances and angle measures are preserved. The image of a region under an isometry is called an ***isometric image*.**

Axiom 4 (Existence of isometries):

i. Given a line, there is one isometry called a ***reflection*** that fixes the line and maps one side of the line to the other side of the line.

ii. Given a ray with endpoint A, and a point B not on the ray, there is an isometry called a ***rotation*** that fixes A and maps the point B to a point on the ray, with the additional property that the angle between $\overrightarrow{AC}$ and its image has the same measure for every point C in the plane.