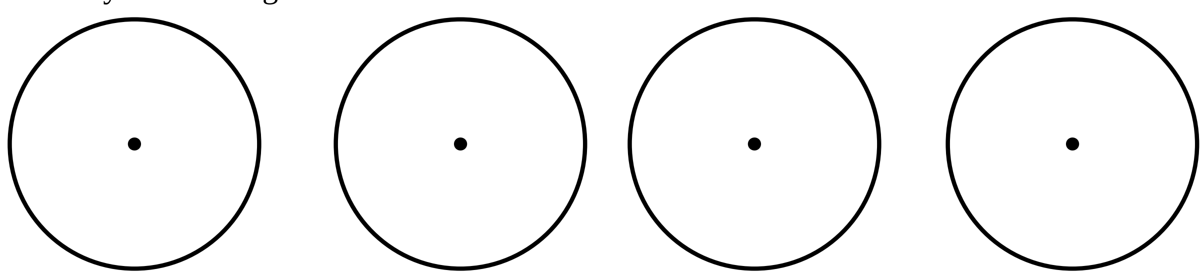
Fractions, Ratios and the Number system:

Models: We're going to use three of the most common models or visualizations for fractions: geometric fractions of a circle and a rectangle, and number line fractions of a length. Each of these models has advantages and disadvantages for learning about fractions. In grade 2, students are introduced to fractions using geometric shapes (such as circles and rectangles) and in grade 3, students are introduced to fractions on a number line.

Most of the visualizations we draw for fractions, are built on an area model of a fraction: the relationship between the area of the part and the area of the whole represents the fraction.

In a **circle** model, the circle represents the whole. The circle is subdivided using radii (segments from the center of the circle to the perimeter), that cut the circle into sectors with equal area. These sectors represent the fractional amount when compared to a whole that is represented by the entire circle.

A circle model of fractions has the advantage that our brains are pretty good at recognizing and estimating angle sizes, so the visualization of the size of a third or a fourth or a fifth is something we're pretty good at remembering and estimating. Practice your estimation strategies on these circles by subdividing them:



thirds fourths fifths sixths

In a **rectangular** model, a rectangle or square (a square is just a special kind of rectangle) represents the whole. The whole is then subdivided using horizontal and vertical lines to make equal sized parts. Rectangles and squares can also be divided into halves or fourths using diagonal lines, but those aren't going to give us the most useful properties that rectangular models have, so we're going to just use subdividing lines that are parallel to the sides.

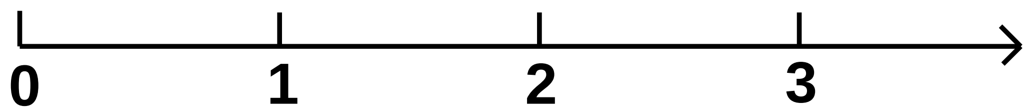
Rectangular models get really useful when you are making subdivisions in both directions at once. For instance, you could subdivide these squares square into sixths in two ways: either by making all of the subdivisions parallel to the same side, or by making some dividing lines vertical and others horizontal.

Show sixths using only vertical lines	Show sixths using some vertical and some horizontal lines	In the pictures of sixths you just drew, can you see thirds in the same picture? Can you see halves?	How would you draw 20ths on a rectangular diagram?

The **number line** model of fractions shows a fraction as a length. The length of a whole unit is marked and labelled as 1 (it is a marked length not a separate shape like a circle or a square). Fraction bars and tape diagrams (which are also called bar diagrams) are variations on the number line model.

Number line fractions are particularly good for representing improper fractions, for comparing fractions to whole numbers, and for thinking of fractions as a kind of number rather than a kind of shape.

On this number line, show the fractions 2/3, and 5/2:



CCSS: In grade 2 children work with simple fractions of geometric shapes:

> CCSS.Math.Content.2.G.A.3 Partition circles and rectangles into two, three, or four equal shares, describe the shares using the words halves, thirds, half of, a third of, etc., and describe the whole as two halves, three thirds, four fourths…

In grade 3, the number line model for fractions is introduced, and is a topic of particular focus:

> CCSS.Math.Content.3.NF.A.2 Understand a fraction as a number on the number line; represent fractions on a number line diagram.

Building on Unit Fractions: Understanding fractions as numbers (grades 3-5)

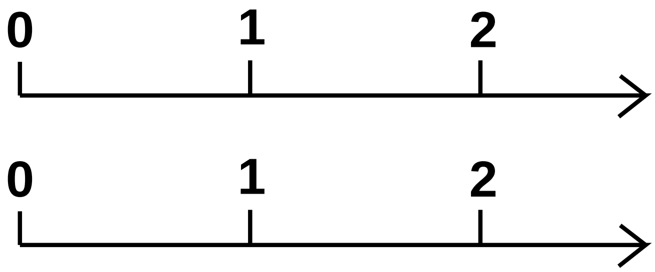
The most important idea for basic fraction knowledge in the common core standards is how to present fractions in terms of unit fractions, as described below:

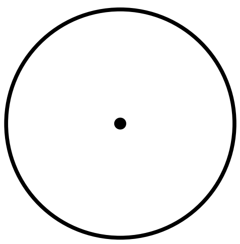
A **unit fraction** is a fraction whose numerator is 1. To show the unit fraction 1/b in any of our models, we start with a representation of 1 whole, and divide it into b equal parts. Each of those parts represents the fraction 1/b. This is always the way that these fractions are defined and presented, so only the name *unit fraction* should be new to you.

Fractions with a numerator greater than 1 are represented and explained as a sum of unit fractions, so 2/3 is 2 units of size 1/3, and 9/4 is 9 units of size ¼. This is slightly different from the most common way of presenting such fractions. The most common way of explaining 2/3 is to say that it is two out of 3 equal parts of a whole. That's a fine explanation for fractions that are less than 1 (proper fractions), but it leads to misunderstandings when children encounter fractions that are greater than 1 (improper fractions). This is the most fundamental change suggested for teaching and understanding fractions in the Common Core Standards compared to the way things have usually been done in the past, and it is a topic of emphasis in grade 3.

Practice: Using a number line and a circle model, show the steps (using unit fractions) to represent the fraction 7/4. Write a sentence explaining each step.

- The first step is to divide the whole unit into equal parts to show the size of the unit fraction (1/4)
- The second step is to draw out the right number of unit fractions to make the fraction you want to show (7/4)



Unit fraction 	
Repeated units:	

CCSS: Understanding fractions as repeated unit fractions is an explicit standard at third grade, and is again reiterated at fourth grade:

> <u>CCSS.Math.Content.3.NF.A.1</u> Understand a fraction $1/b$ as the quantity formed by 1 part when a whole is partitioned into *b* equal parts; understand a fraction a/b as the quantity formed by a parts of size $1/b$.
>
> <u>CCSS.Math.Content.4.NF.B.3</u> Understand a fraction a/b with $a > 1$ as a sum of fractions $1/b$.

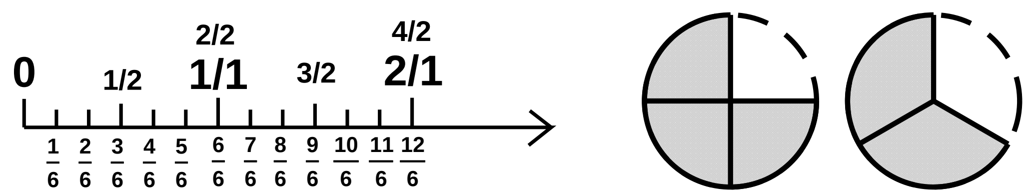
Fraction Equivalence and Comparing Fractions:

The teaching of fraction equivalence and fraction comparison is divided into two stages: a visual reasoning stage and a computational reasoning stage.

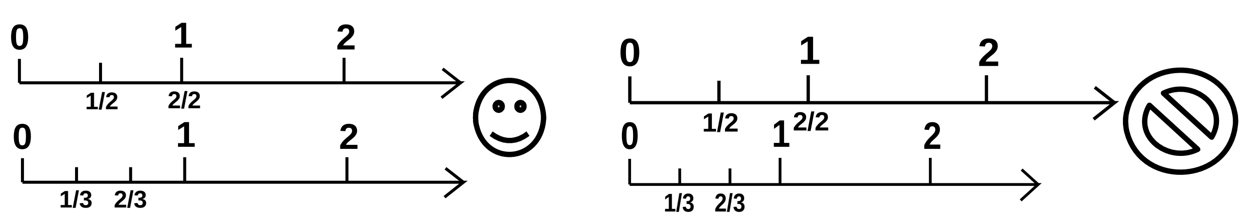
- In the *visual reasoning* stage, the main concepts are that the sizes of fractions can be compared using manipulatives and diagrams and by reasoning directly from the size of unit fractions. (grades 2-3 in CCSS)
- In the *computational* stage, students learn to generate equivalent fractions using multiplication and fractions are compared by finding a common denominator (usually) or a common numerator (occasionally). (grade 4 and up)

A **visual** approach to **equivalent fractions** is to say that *two fractions are equivalent if they show the same part of the whole*. In an area model (circles or rectangles), that means that the fractions cover the same area. In a number line (length) model, that means the fractions are at the same point on the number line or show the same length. Reasoning about this equivalence is often done by comparing physical models. It's easy to make mistakes with pictures and manipulatives if the unit fractions are very small, but experience with pictures and manipulatives helps children develop visualization and estimation skills.

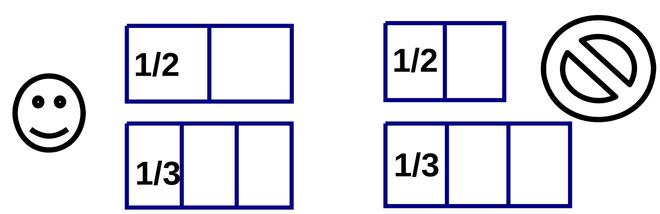
Another detail to notice is that the notation for writing a whole number as a fraction (2=2/1) is something that is now introduced in third grade along with putting fractions on a number line.



A **visual** approach to **comparing fractions** is to show that one fraction is larger than another if it shows a larger part of the same whole. In an area model, the area is larger, and in a length model the length is longer. The concept that *the size of the whole must be the same in order to compare fractions* is a key goal of instruction at this level. For example, to compare two fractions on number lines, the length of the whole unit 1 must be the same on both number lines:



The size of the rectangle that shows 1 must be the same for both fractions in a rectangular model:



CCSS grade 3

CCSS.Math.Content.3.NF.A.3 Explain equivalence of fractions in special cases, and compare fractions by reasoning about their size.

CCSS.Math.Content.3.NF.A.3a Understand two fractions as equivalent (equal) if they are the same size, or the same point on a number line.

CCSS.Math.Content.3.NF.A.3b Recognize and generate simple equivalent fractions… . Explain why the fractions are equivalent, e.g., by using a visual fraction model.

CCSS.Math.Content.3.NF.A.3c Express whole numbers as fractions, and recognize fractions that are equivalent to whole numbers. *Examples: Express 3 in the form 3 = 3/1; recognize that 6/1 = 6; locate 4/4 and 1 at the same point of a number line diagram*.

CCSS.Math.Content.3.NF.A.3d Compare two fractions with the same numerator or the same denominator by reasoning about their size. Recognize that comparisons are valid only when the two fractions refer to the same whole. Record the results of comparisons with the symbols >, =, or <, and justify the conclusions, e.g., by using a visual fraction model.

Reasoning with unit fractions can explain some comparisons without relying on a picture and also without computing common denominators. This kind of reasoning builds a good foundation for future fraction learning. The two most important examples of this are comparisons when either the denominators are already the same or the numerators are already the same in the two fractions.

- If the denominators are the same, then the fractions are composed of *unit fractions of the same size*, so the fraction with more units is larger.
- If the numerators are the same, then the fractions are composed of the same number of unit fractions, so we should compare the size of the unit fractions. Remember that unit fractions are made by dividing a whole into a given number of equal sized pieces. *If the whole is divided into more pieces, each unit fraction must be smaller*. The fraction with the larger denominator is composed of smaller unit fractions, so its size must be smaller.

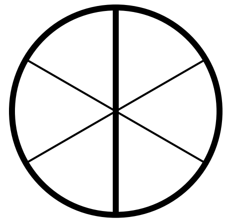
Explaining comparisons using unit fractions is probably new to you. Practice with these pairs of fractions:

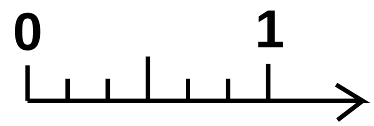
Explain: • Are the unit fractions the same? • If the unit fractions are not the same, which is larger? Why? • Are there the same number of unit fractions? • How does the number of unit fractions and the size of unit fractions tell you which fraction shows the larger amount?	Which is larger, 7/15 or 8/15?	Which is larger, 5/8 or 5/9?

Learning to draw fractions whose denominator is a product can help you find **equivalent fractions by reasoning about size**. A good way to draw fractions whose denominator is a product is to *split* the whole into the fraction given by one factor, and then *split* each part into the number of parts given by the other factor. For example:

TIP: By making the dividing lines different colors, you can make it easier to explain to others

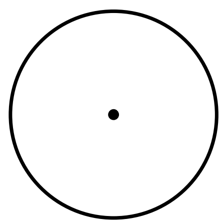
To draw sixths: $\frac{1}{2\times 3}$
First draw in halves.
Then split each half into 3 equal parts





You can use this process to find and explain some equivalent fractions. For example, if you make sixths by first making halves, you can say:

I split 1/2 into 3 parts, which split the whole into 6 parts, and each part is 1/6 that means 1/2 is 3 units of size 1/6, so 1/2=3/6.

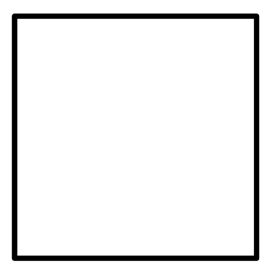
You try it the other way: To draw sixths: $\frac{1}{2\times 3}$ First draw in thirds. Then split each third into 2 equal parts		

Now explain the equivalent fraction (1/3=…)

In a rectangular model, you can make subdividing lines vertically for one of the factors, and horizontally for the other factor:

To draw sixths: $\frac{1}{2\times 3}$

- Draw 1 vertical line to make halves
- Draw 2 horizontal lines to split each half into 3 equal parts



The most important fraction ideas in grade 3 are:

- Fractions are made up of (repeated) unit fractions (the number of unit fractions is given by the number in the numerator)
- The size of a unit fraction is determined by the whole amount you start with, and you need to divide the whole into equal sized parts (the number of parts is given by the number in the denominator)

CCSS grade 4:

CCSS.Math.Content.4.NF.A.1 Explain why a fraction *a*/*b* is equivalent to a fraction (*n* × *a*)/(*n*× *b*) by using visual fraction models, with attention to how the number and size of the parts differ even though the two fractions themselves are the same size. Use this principle to recognize and generate equivalent fractions.

A **computational** approach to equivalent fractions is what you probably remember from learning fractions in elementary and middle school. The computational approach is/should be introduced *after* children have spent time developing concepts and intuition using manipulatives, drawings and reasoning from unit fraction size.

In the computational approach to equivalent fractions, we create equivalent fractions using multiplication and division. The computational approach is/should be introduced through pictures that children are familiar with using to represent fractions.

The split and split again strategy (split using the first factor and them split each part using the second factor) for drawing fractions with denominators that are a product of two numbers is a good introduction to the thinking that you need to build a computational strategy for finding common denominators.

Getting the formula $\frac{a}{b} = \frac{n \times a}{n \times b}$ from a reasoning about unit fractions with a picture has a denominator and a numerator step. This is an outline of the reasoning (using variable *a* and *b*).

- Unit fractions (denominator): A unit fraction $1/b$ is one of *b* equal parts of the whole. If you take that unit fraction and split it into *n* equal parts, then there are $n \times b$ equal parts of that size in the whole. Each part is a unit fraction $\frac{1}{n \times b}$.
- Number of unit fractions (numerator): If the fraction is a/b then there are *a* parts of size $1/b$. If you split those *a* parts into *n* equal parts each, you will have $n \times a$ parts of size $\frac{1}{n \times b}$ so $\frac{a}{b} = \frac{n \times a}{n \times b}$

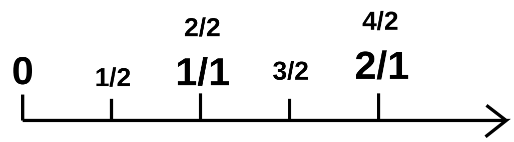
When you're talking about this process with fourth graders, the best way to do this is to explain is with numbers (not variables) and to talk about the same process several times with different numbers, and to notice what steps are the same each time. When you've talked about why it works with specific numbers, then you can write down the formula with variables: seeing letters in a formula is a good way to get used to the idea of using letters to stand for numbers.

Examples of explanations with specific numbers and pictures: Some key ideas and steps:

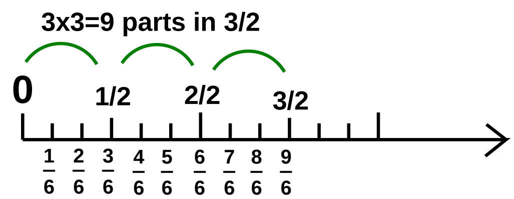
- The visual model shows that the fractions are equivalent because they are *the same size*.
- I use the model and the fraction explanations to explain the computational strategy, so that *the numbers come from the diagram*, not vice versa.
- I explain the denominator and the numerator separately and then put them together
- I use multiplication to find the number of parts at each step.

Explaining equivalent fractions with a number line:

We're looking for a fraction equivalent to 3/2



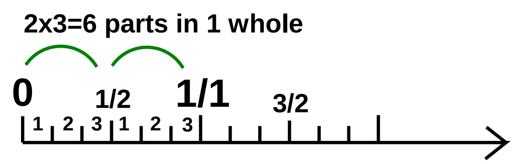
Take each half and split it into 3 equal parts



There are 3 small parts in each of the 3 halves that make up 3/2. So there are 3 groups of 3 $= 3\times 3 = 9$ parts in 3/2.

$$\frac{3}{2} = \frac{3\times 3}{3\times 2} = \frac{9}{6}$$

There are 3 small parts in each of the 2 halves in a whole. So there are 2 groups of 3 = $2\times 3 = 6$ equal parts in a whole.

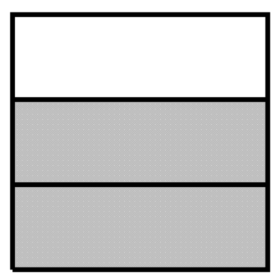


(each of these smaller parts are $\frac{1}{2\times 3} = \frac{1}{6}$)

Notice that I'm writing out the answer using the factors: 3×3 and 3×2. That's me showing the computation strategy, which is: if you multiply the numerator and denominator by the same factor, you get an equivalent fraction.

Explaining equivalent fractions in a rectangular model

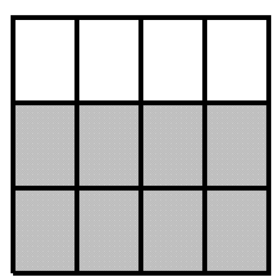
We're looking for a fraction equivalent to 2/3. We represent 2/3 using horizontal subdivisions.



We split each of the 3 thirds into 4 equal parts by using a vertical subdivision

Each of the 2 shaded thirds were split into 4 smaller parts so there are 2 groups of 4 = $2\times 4 = 8$ smaller shaded parts.

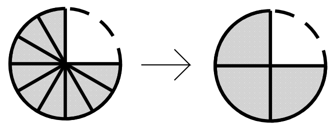
$$\frac{2}{3} = \frac{2\times 4}{3\times 4} = \frac{8}{12}$$



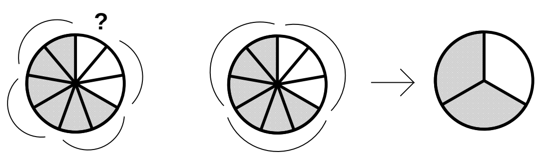
Each of the 3 thirds in a whole were split into 4 smaller parts. There are 3 groups of 4 = $3\times 4 = 12$ smaller parts in the whole (each part has size $\frac{1}{12}$)

Simplifying fractions:

Visually simplifying a fraction means that you are looking for larger sized unit fractions to make the same amount. With manipulatives that can mean guessing and checking with larger sized fraction pieces to see if any fit perfectly.



With a diagram, to get a larger sized unit fraction, you need to put together smaller unit fractions to make a larger one—the trick is that you have to be able to group all of the unit pieces in your fraction *and* you need to be able to group all of the unit pieces in a whole using the same grouping strategy



The 6 ninths in 6/9 can be put into groups of 2 or groups of 3, but the 9 ninths in a whole can only be put into groups of 3, so the only way to simplify the fraction is to make groups of 3. Making groups of 3 corresponds to dividing by 3, so $\frac{6}{9}=\frac{6\div 3}{9\div 3}=\frac{2}{3}$

Numerically simplifying a fraction is the process of going backwards from our earlier process of finding an equivalent fraction: the process of finding an equivalent fraction with a *larger denominator* (smaller unit fraction) is to multiply the numerator and denominator by the same number:

$$\frac{a}{b}=\frac{n\times a}{n\times b} \qquad\qquad \frac{2}{3}=\frac{2\times 5}{3\times 5}=\frac{10}{15}$$

Simplifying a fraction is a process that finds an equivalent fraction with a *smaller* denominator (larger unit fraction). This process will un-do the previous process where we multiply by the same factor in the numerator and denominator.

There are two ways of thinking of a backwards multiplication, one is division, and the other is factoring. When you simplify a fraction, you can write the process as dividing or as factoring:

Divide: $$\frac{a\times n\div n}{b\times n\div n}=\frac{a}{b} \qquad \frac{10}{15}=\frac{10\div 5}{15\div 5}=\frac{2}{3} \text{ or } \frac{10}{15}=\frac{10}{15}\div\frac{5}{5}=\frac{2}{3}$$

Factor: $$\frac{a\times \cancel{n}}{b\times \cancel{n}}=\frac{a}{b} \qquad \frac{10}{15}=\frac{2\times \cancel{5}}{3\times \cancel{5}}=\frac{2}{3}$$

You can also figure out how to do all of the simplifying in one step, or you can do several smaller steps:

several steps | one step

divide $$\frac{60}{72}=\frac{60\div 2}{72\div 2}=\frac{30\div 2}{36\div 2}=\frac{15\div 3}{18\div 3}=\frac{5}{6} \qquad \frac{60}{72}=\frac{60\div 12}{72\div 12}$$

factor $$\frac{60}{72}=\frac{30\cancel{\times 2}}{36\cancel{\times 2}}=\frac{15\cancel{\times 2}}{18\cancel{\times 2}}=\frac{5\times 3}{6\times 3}=\frac{5}{6} \qquad \frac{60}{72}=\frac{\cancel{2}\times\cancel{2}\times\cancel{3}\times 5}{\cancel{2}\times\cancel{2}\times 2\times\cancel{3}\times 3}=\frac{5}{6}$$

All of these strategies are commonly taught (different ones by different teachers and textbooks).

Division: Probably the most common strategy to simplify fractions is to *divide* the numerator and denominator in *several steps*. Some advantages of this strategy are:

- We learn very early on to think of division as the opposite of multiplication. This makes the division notation make sense to us, and it fits that it would un-do what the multiplication process does.
- By simplifying a bit at a time, we can immediately use any common factors we notice to make the rest of the problem easier.

Since this is a very common strategy, I want to point out two good ways of writing the process down (and one I want you to avoid)

Good, correct, notation #1 (most common)	Correct notation #2 (less standard)	A notation that's wrong
$\frac{12\div 2}{18\div 2}=\frac{6\div 3}{9\div 3}=\frac{2}{3}$	$\frac{12}{18}\div\frac{2}{2}=\frac{6}{9}\div\frac{3}{3}=\frac{2}{3}$	$\frac{12}{18}\div 2=\frac{6}{9}\div 3=\frac{2}{3}$ (because the notation says to divide the whole fraction by 2, not divide the numerator and the denominator each by 2)

Some people don't like doing lots of little steps—they feel uncomfortable because they worry that they might miss a factor and not simplify enough. If you want to do all of the division in one step, you first need to find the GCF (greatest common factor). Finding the GCF can sometimes take a lot of steps, so this is usually a more complicated process.

Teachers and textbooks that emphasize finding the greatest common factor often put in a separate step in the process just for finding the GCF.	$\frac{32\div 16}{48\div 16}=\frac{2}{3}$ Factors of 32: 1, 2, 4, 8, 16, 32 Factors of 48: 1, 2, 3, 4, 6, 8, 12, 16, 24, 48

Factoring seems to be a little less common way of teaching simplifying fractions, but it's a very powerful tool to learn how to use, especially when using a prime factorization. Some advantages of finding a prime factorization are:

- The simplifying part of the problem happens all in one step
- It prepares your brain for using factoring to find least common denominators when adding fractions
- It prepares your brain to simplify algebraic fractions by factoring when you learn about them in Algebra 2.

Notice that most of these advantages have to do with building good habits of thinking about fractions that will help with later (harder) work.

Simplifying using prime factorizations works by:

Finding the prime factorizations of the numerator and denominator	Writing out the fraction in factored form, and cancelling common factors:	I combine these when I teach it:
42 → 6, 7; 6 → 2, 3 140 → 14, 10; 14 → 2, 7; 10 → 2, 5	$\frac{42}{140}=\frac{\cancel{2}\times 3\times\cancel{7}}{\cancel{2}\times 2\times 5\times\cancel{7}}=\frac{3}{10}$	$\cancel{2}$ 3 $\cancel{6}$ $\cancel{7}$ $\frac{\cancel{42}}{\cancel{140}}=\frac{3}{10}$ $\cancel{10}$ $\cancel{14}$ $\cancel{2}$ 5 2 $\cancel{7}$

Common denominators

CCSS Grade 4:

CCSS.Math.Content.4.NF.A.2 Compare two fractions with different numerators and different denominators, e.g., by creating common denominators or numerators, or by comparing to a benchmark fraction such as 1/2. Recognize that comparisons are valid only when the two fractions refer to the same whole. …

CCSS Grade 5:

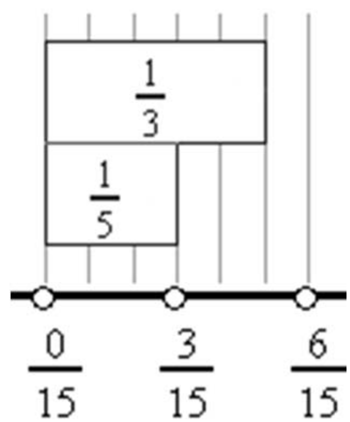
CCSS.Math.Content.5.NF.A.1 Add and subtract fractions with unlike denominators (including mixed numbers) by [finding a common denominator]

CCSS.Math.Content.5.NF.A.2 Solve word problems involving addition and subtraction of fractions referring to the same whole, including cases of unlike denominators…. Use benchmark fractions and number sense of fractions to estimate mentally and assess the reasonableness of answers. …

Common denominators: what are they? Why do we need them?

A fraction's size is determined by two things: its denominator, which determines the size of each unit, and its numerator, which determines how many units it is made of. When we have two fractions to compare, it's easy to say which has the most units, and which has the largest units, but unless one of those number is the same for both fractions, it's sometimes hard to know which is bigger: the one with a lot of little units, or the one with a few big units? To compare the fractions, we need a way to say *how much* bigger one unit fraction is than another. A common denominator lets us say how large each of the unit fractions are as measured in a common unit.

Think of a unit fraction as being a unit like inches and centimeters: inches and centimeters measure lengths and unit fractions measure numbers.

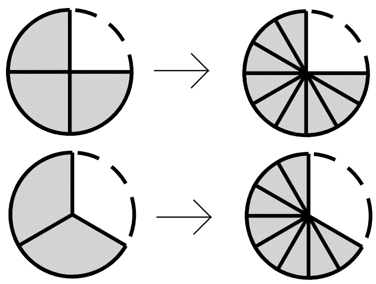


The unit fraction 1/15 is the right size to measure both fifths and thirds. That means that 1/3 is a whole number of fifteenths and 1/5 is a whole number of fifteenths. Here, the length of 1/5 is the same length as 3/15 and 1/3 is the same length as 5/15. When we say we have found a common denominator, that's the same as saying we've found a unit fraction that's the right size to measure both of the other fractions and get whole number amounts of the same unit fraction.

When we are comparing two fractions, we compare fractional amounts *of the same whole*. When we add or subtract two fractions we put together, take apart, or find the difference between fractional amounts *of the same whole*. When both amounts are fractions *of the same whole*, we can make them easier to work with if we find equivalent fractions that have the same sized unit fraction: a common denominator.

Finding common denominators:

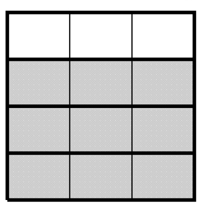
Visually: Finding a common denominator visually with fraction manipulatives is essentially a guess and check process. You try building the unit fractions you start with out of smaller unit fractions, trying one and then another until you find one that works for both of your starting units.



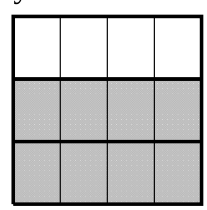
$$\frac{3}{4}=\frac{9}{12}$$

$$\frac{2}{3}=\frac{8}{12}$$

If you're trying to accomplish the same thing when drawing fractions, you'll usually end up creating the common denominator that is the product of the two denominators you started with. You can do this by using the split and split again process for making fractions, using the two denominators (this looks particularly symmetric if you draw it with squares):



Split each of the fourths in 3/4 into 3 pieces to get $\frac{3}{4}=\frac{9}{12}$



Split each of the thirds in 2/3 into 4 pieces to get $\frac{2}{3}=\frac{8}{12}$

Numerically: Finding common denominators numerically is often equated with finding the *least* common denominator, so the first thing to realize is that you *can* solve problems by finding any common denominator, even if it isn't the *least* common denominator. The main advantage that a least common denominator solution has is that (when you are adding and subtracting) there is less simplifying to do at the end.

Example of solving with a *least* common denominator:

$$\frac{3}{4}+\frac{5}{6}=\frac{3\times 3}{4\times 3}+\frac{5\times 2}{6\times 2}=\frac{9}{12}+\frac{10}{12}=\frac{19}{12}=1\frac{7}{12}$$

Example of solving with the *product of the denominators*:

$$\frac{3}{4}+\frac{5}{6}=\frac{3\times 6}{4\times 6}+\frac{5\times 4}{6\times 4}=\frac{18}{24}+\frac{20}{24}=\frac{38}{24}=\frac{19\times \cancel{2}}{12\times \cancel{2}}=\frac{19}{12}=1\frac{7}{12}$$

One implication of this is that there may be times when you want students to find a common denominator, and you don't insist that it be the smallest one possible. Presenting the question as: "What is a common denominator that would work to add $\frac{3}{4}+\frac{5}{6}$?" allows for the easier solution of $4\times 6=24$ as well as the smallest solution (12), which requires a better knowledge of multiplication and division facts.

Least Common Denominators: Multiples and factors.

Multiples: The least common denominator is the smallest number that is evenly divisible by both denominators. You can use multiples of the denominators to find the least common multiple. We can do this by writing out lists of the multiples (or equivalent fractions) for both fractions, and then looking for the first common multiple (or common denominator) on the list.

Two lists: For example, with the fractions $\frac{4}{9}$ and $\frac{8}{21}$, making lists might look like this:

Listing multiples of the denominators:	Listing equivalent fractions:
$9, 18, 27, 36, 45, 54, 63, 72, 81$ $21, 42, 63$	$\frac{4}{9}, \frac{8}{18}, \frac{12}{27}, \frac{16}{36}, \frac{20}{45}, \frac{24}{54}, \boxed{\frac{28}{63}}, \frac{32}{72}, \frac{36}{81}$ $\frac{8}{21}, \frac{16}{42}, \boxed{\frac{24}{63}}$

Most people just make the lists with the denominators, but making lists with equivalent fractions as a first introduction helps keep the focus on the goal: finding equivalent fractions with the same denominator.

One list: A slightly more efficient variation on making lists is to find the multiples of just the larger denominator, and at every step, check and see if that number is a multiple of the smaller denominator too:

$21\times1=21$ Is 21 a multiple of 9? No	$21\times2=42$ Is 42 a multiple of 9? No	$21\times3=63$ Is 63 a multiple of 9? Yes: $9\times7=63$

Using the LCM: After you find the least common multiple, the next goal is to find the equivalent fractions with a common denominator:

$\frac{4\times7}{9\times7}=\frac{28}{63}$ and $\frac{8\times3}{21\times3}=\frac{24}{63}$

Factors: The common denominators are always going to be products (the result of multiplying is a product) of both numbers. Any number is the product of its factors. If we find the factors of a number, we can multiply the factors together to get the least common denominator.

Prime factors: One good way to get the right factors (and therefore get the *least* common denominator) is to find the prime factorization of both denominators:

$\frac{5}{9}$ $\frac{8}{21}$ / \ / \ 3 3 3 7	Factor 9 to get 3×3 Factor 21 to get 3×7 I circle the factors they have that are the same, and underline the factors that one has and the other doesn't.	$\frac{7}{7}\cdot\frac{5}{9}$ $\frac{8}{21}\cdot\frac{3}{3}$ / \ / \ 3 3 3 7 The common denominator is $3\times3\times7=63$	Multiply the fraction on the left by the factor (7) that the denominator on the right has (and it doesn't). Multiply the fraction on the right by the factor (the second 3) that the fraction on the left has and it doesn't.

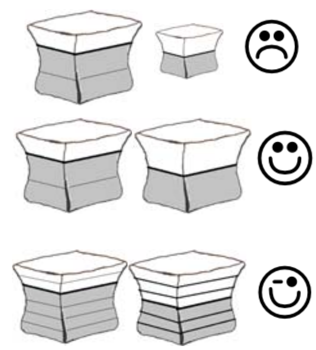
Why do you need common denominators to add and subtract?

A common way of thinking about **addition** is that it is putting together two groups, and finding out how many in all. The things you put together have to be the same kinds of things—things that are counted in the same way. For example, you could have:

- 3 apples and 2 more apples: then you would have 5 apples in all
- 3 students and 2 teachers (which is 3 people and 2 more people), so you would have 5 people in all. Notice that I have to think about both the students and teachers as *people* for the question "how many in all" to make sense.

When we add with fractions, we first need the unit wholes to be the same size, and then we need the unit fractions to be the same size:

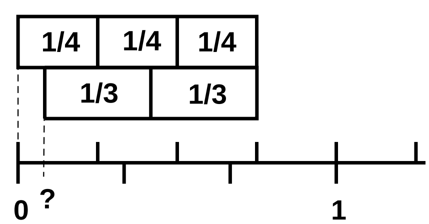
- If you have 1/2 of a bag of beans and 2/3 of a bag of beans, then in order to figure out how many beans there are, we first need to have a problem where the *full bags* would have the same amount in them. Next, we need to know that 1/2 bag is 3 times as large as 1/6 bag (1/2 = 3/6), and 2/3 bag is 4 times as large as 1/6 bag (2/3 = 4/6). At that point we have figured out how to think of the two smaller groups as being counted in the same way: in 1/6 bag units. Notice that the total consists of counting/adding the 1/6's together: 3 pieces of size 1/6 + 4 pieces of size 1/6 = 7 pieces of size 1/6.



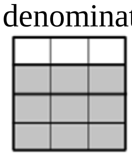
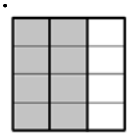
Just like I needed to think of students and teachers as both being people in the previous problem, with fractions, I need to think of halves and thirds as both being made out of sixths.

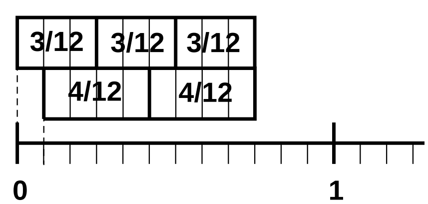
Subtraction can be thought of as the process of taking some out of a larger group, or comparing two groups to find out how much larger one is than the other.

This number line shows 3/4 and 2/3 lined up so you can see how much bigger 3/4 is than 2/3. The picture shows how much more is in a 3/4 lb bag of beans than a 2/3 lb. bag of beans. The picture doesn't tell me, though, just how much that amount is—to do that I need to find a common denominator.



I can split up both fourths and thirds into twelfths—12 is the common denominator.

$\frac{3}{4}=\frac{9}{12}$   $\frac{2}{3}=\frac{8}{12}$



When both amounts are expressed in terms of the same unit fractions, then whole number subtraction (with the numerators) will tell us the exact difference. There is one more piece of size 1/12 in 9/12 than in 8/12: $\frac{9}{12}-\frac{8}{12}=\frac{1}{12}$

Multiplying is different:

Adding two numbers tells how many there are when you put together two sets. **Subtracting** tells how many when you take away or separate out part from a set *or* how many more are in one set than another. In an addition or subtraction word problem, the sets of things you are putting together, taking apart or comparing have to be the same kinds of things: you have to be able to count or measurement them in the same way using the same units

Take away 4 *pencils* from a set of 12 *pencils* and find out how many are left.
Put together 4 apples and 5 oranges, and find how many *pieces of fruit* in all.
Compare 5 *lbs* of flour and 3 *lbs* of sugar, and find how many more *pounds* of flour there are than sugar.

Multiplying is different. The most common way of thinking of multiplication is repeated groups: in multiplying, we are finding how many in a number of groups of things. The two numbers in a multiplication sentence represent different things: one represents a ***number of sets***, and the other a ***number of things in a set***. So in a multiplication problem you might have:

There are 2 boxes and 5 pencils in each box, and find how many pencils in all.
There are 6 cookies, and each cookie weighs 2 ounces, find how many ounces they weigh together.

Because you need to measure the things you are **adding or subtracting** in the same way, it's necessary for *the things you are adding to be fractions of the same size whole*, and to find the answer you need a *common denominator* so that both things are being measured or counted using the same unit fraction:

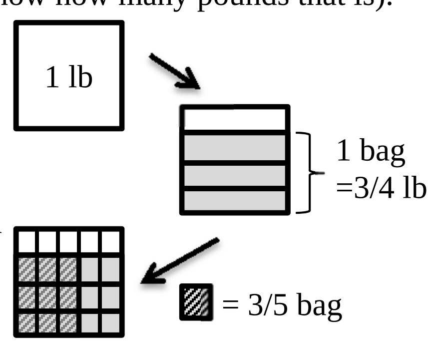
Because the numbers in a **multiplication** problem are *amounts* and *sets* (not amounts and amounts), it isn't helpful to have a common denominator when multiplying fractions. Instead, multiplying fractions involves *subdividing the fractional amount using the other fractional number of sets*.

A bag of candy that holds 3/4 lb. of candy is 3/5 full. How much candy is there?

- The amount in a whole set (a whole bag) is 3/4 lb
- The number of sets is 3/5 (of a bag)

At the end we want an answer that is a number of pounds (we already know there's 3/5 of a bag, so that isn't what we're looking for—we want to know how many pounds that is).

- That means we should start with a diagram that shows pounds.
- Next we show the size of a whole bag (3/4 lb.)
- Then we split the part that shows the bag into fifths, and show 3/5 of the bag. (Split the other fourth-pounds into fifths too so it's easier to find the final answer)

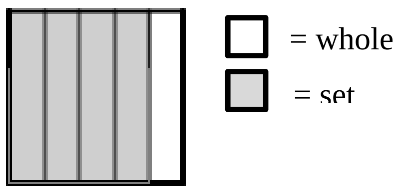


To find the answer, we figure out what size each of the pieces are. There are 4×5 parts in 1 lb., so each piece represents 1/20 lb.. There are 3×3 parts in the 3/5 bag we have, so we have 9/20 lb. of candy.

Explaining the Multiplication of Fractions Process

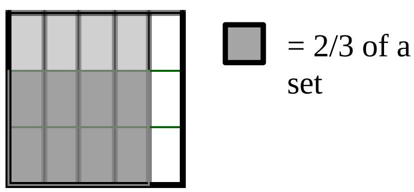
Part 1: What does it mean?
With whole numbers: 3×4 means the amount in 3 sets of size 4.
So, with fractions $\frac{2}{3}\times\frac{4}{5}$ means the amount in $\frac{2}{3}$ of a set of size $\frac{4}{5}$
Part 2: Make a diagram:

- Show the size of one set, which is 4/5 of a whole



Tip: I chose some good numbers: all the numbers are different, and nothing will cancel out.
I'm also going to using a rectangular diagram, which is going to be the easiest one to explain the product with.

- Show what 2/3 of the set is



- **Part 3: Use the diagram to find the answer as a fraction of the whole. Connect the answer to the diagram and the original numbers.**

The whole has 3×5 equal sized parts: 3 down by 5 across.
The dark shaded part (2/3 of 4/5) has 2×4 parts: 2 down by 4 across.
So 2/3 of 4/5 is $\frac{2\times 4}{3\times 5}=\frac{8}{15}$ of the whole
(Writing it out with the multiplications is important for showing the patterns of how to get the product numerically—don't skip that step).

This shows how we get the **numerical algorithm** for multiplying $\frac{a}{b}\times\frac{c}{d}=\frac{a\times c}{b\times d}$ from the reasoning with a diagram about what multiplication of fractions should mean.