$Q[i] = \{a+bi\} = Q(i) \leftarrow$ field

$\leftarrow$ let $a+bi \in Q[i]$ and $a+bi \neq 0$

What is $(a+bi)^{-1}$?

$$(a+bi) = \frac{1}{(a+bi)} \cdot \frac{(a-bi)}{(a-bi)} = \frac{a-bi}{a^2-b^2i^2} = \frac{a-bi}{a^2+b^2}$$

$$= \boxed{\frac{a}{a^2+b^2}} + \boxed{\frac{-bi}{a^2+b^2}} \in Q[i]$$

so $Q[i]$ is a field

at least one of a & b is not 0.
so a^2+b^2 is not 0.

$$(a+b\sqrt{2})^{-1} = \frac{1}{a+b\sqrt{2}} \cdot \frac{a-b\sqrt{2}}{a-b\sqrt{2}} =$$

$$\frac{a-b\sqrt{2}}{a^2-b^2\cdot\sqrt{2}^2} = \frac{a-b\sqrt{2}}{a^2-2b^2}$$

Suppose a, b are rational and not both 0.

suppose $a^2 - 2b^2 = 0$

$$a^2 = 2b^2$$

Case 1: $b = 0$

then $a = 0$

contradiction

Case 2: $b \neq 0$

$$2 = \frac{a^2}{b^2}$$

$$\sqrt{2} = \frac{a}{b}$$

False because $\sqrt{2}$ is irrational

contradiction

so $a^2 - 2b^2 \neq 0$