Functions on General Sets

$f : \text{domain} \to \text{codomain}$ A function takes objects in one set and maps them to objects in another set. The starting set is the **domain**: the function needs to map *everything* in the domain to *one and only one thing* in the ending set. The ending set is the **codomain**: all of the things that are mapped to have to be in the codomain. A codomain is a bit like a range, except that it's OK if not everything in the codomain is mapped to. In algebra, almost all of the functions you see have the real numbers as both the domain and the codomain.

Examples of functions from algebra:

$f(x) = 2x$ domain: $\mathbb{R}$ range: $\mathbb{R}$ codomain $\mathbb{R}$ $f : \mathbb{R} \to \mathbb{R}$	$g(x) = x^2$ domain: $\mathbb{R}$ range: $[0,\infty) = \{x \geq 0 \mid x \in \mathbb{R}\}$ codomain $\mathbb{R}$ or $[0,\infty)$ $g : \mathbb{R} \to \mathbb{R}$	$h(x) = {}^{1}\!/_{x}$ domain: $\{x \mid x \in \mathbb{R}, x \neq 0\}$ range: $\{x \mid x \in \mathbb{R}, x \neq 0\}$, codomain $\mathbb{R}$ (or range) $h : \{x \mid x \in \mathbb{R}, x \neq 0\} \to \mathbb{R}$

When working with general sets, it's important to specify the domain and codomain.

Number set examples:

$f(x,y) = x + 2y$ domain: $\mathbb{R} \times \mathbb{R}$ codomain: $\mathbb{R}$ $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$	$g(z) = z^2$ domain: $\mathbb{C}$ codomain: $\mathbb{C}$ $g : \mathbb{C} \to \mathbb{C}$	$h(a+bi) = \sqrt{a^2 + b^2}$ domain: $\mathbb{C}$ codomain: $\mathbb{R}$ or $[0,\infty)$ $h : \mathbb{C} \to [0,\infty)$

Geometry examples:

$f : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ such that $f(x,y) = (x+3, y)$ This function translates all of the points in the plane to the right 3 units.	$S^1 = \{(x,y) \mid x^2 + y^2 = 1\}$ (unit circle) $Q = \{(x,y) \mid x = \pm 1 \text{ or } y = \pm 1\}$ (square) $g : Q \to S^1$ such that $g(x,y) = \left(\dfrac{x}{\sqrt{x^2+y^2}}, \dfrac{y}{\sqrt{x^2+y^2}} \right)$

Less familiar examples:

$C^1(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is differentiable on } \mathbb{R}\}$ is a set, and its elements are differentiable functions. $F(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R}\}$ is the set of functions on $\mathbb{R}$ $\dfrac{d}{dx} : C^1(\mathbb{R}) \to C(\mathbb{R})$ such that $\dfrac{d}{dx}(f) = f'$ Taking a derivative maps a (differentiable) function to another function.	$Cir = \{\{(x,y) \mid x^2 + y^2 = r^2\} \mid r \in \mathbb{R}\}$ is the set of circles with center at the origin. It's elements are circles: $\{(x,y) \mid x^2 + y^2 = r^2\}$ $f : \mathbb{R} \to Cir$ such that $f(r) = \{(x,y) \mid x^2 + y^2 = r^2\}$ is a function that maps a radius to a circle: it's a function, because the output isn't one or another point, it's the (single) circle.

A programming example (outline of defining a class and method)

public class warranty{
    String serialNumber;
    Date purchaseDate;
    long warrantyLength;
  public int timeLeft(){
    now = Date();
    elapsed = now – purchaseDate;
    elapsed = max(0,warrantyLength - elapsed);
    return elapsed; } }
	This class defines a new kind of object called a warranty, with a specified serial number, date of purchase, and the length of the warranty.
	timeLeft() is a method that takes a warranty object, and tells how much time is left on the warranty.
	The method is a function whose *domain* is all of the objects in the class warranty, and whose *codomain* is a number which specifies how much time is remaining in the warranty.

More function vocabulary:

Mapping: A function is also called a **mapping**. For the function $f:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ such that $f(x,y)=xy$, we can say f is a **mapping** and we say f **maps** $\mathbb{R}\times\mathbb{R}$ to $\mathbb{R}$, and f **maps** (x,y) to xy

Image: If $f:D\to C$ is a function, and $x\in D$ is an element the domain, then $f(x)$ is the **image** of x.

- If S is a subset of the domain then the set of all images of elements of S is the image of S: the **image** of S. There are two notations for this: $f(S)=f^{\rightarrow}(S)=\{f(x)\mid x\in S\}$

Pre-Image: If $f:D\to C$ is a function, and $y\in C$ is in the codomain, then $\{x\mid f(x)=y\}$ is the **pre-image** of y. Note that this is a set, not just an element because several elements in the domain could map to y.

- If T is a subset of the codomain then the set of all elements that map to an element of T is the **pre-image** of T. There are two notations for this set: $f^{-1}(T)=f^{\leftarrow}(T)=\{x\mid f(x)\in T\}$
  Note 1: a pre-image will be the empty set if the set or element is not in the range
  Note 2: the first notation $f^{-1}(T)$ is more common, but we don't know that f^{-1} is a function, so it's possible to get confused between the notation $f^{-1}:C\to D$ for an inverse function, and the notation $f^{-1}:C\to 2^D$, where an element in maps to the a set of elements in D.

Practice Problems:

Here are a few functions. Describe the images and pre-images that each question asks for:

1. $f:\mathbb{R}^2\to\mathbb{R}^2$ such that $f(x,y)=(-y,x)$

Find the images of: a. (3,5) b. $\{(x,y)\mid y=2x\}$ (graph the set and the image in different colors)

Find the pre-images of: c. (2, 6) d. $\{(x,y)\mid y=x^2\}$ (graph the set and the pre-image)

2. $f:\mathbb{R}^2\to\mathbb{R}^2$ such that $f(x,y)=(x+3,y-1)$

a. graph (4, 2) and $f(4,2)$ b. graph $\{(x,y)\mid y=|x|\}$ and $f^{\rightarrow}(\{(x,y)\mid y=|x|\}$

c. graph (5,0) and $f^{\leftarrow}(5,0)$ d. graph $\{(x,y)\mid x^2+y^2=4\}$ and $f^{\leftarrow}(\{(x,y)\mid x^2+y^2=4\})$

3. $g:\mathbb{Z}\to\mathbb{Z}$ such that $g(x)=4x$

Find the images of a. 10 b. $3\mathbb{Z}=\{3n\mid n\in\mathbb{Z}\}$

Find the pre-images of c. 8 d. 10 e. $3\mathbb{Z}=\{3n\mid n\in\mathbb{Z}\}$

4. $h:\mathbb{Z}\times\mathbb{Z}\to\mathbb{Z}$ such that $h(x,y)=xy$

a. $h(2,3)$ b. $h(\{(2,x)\mid x\in\mathbb{Z}\})$ c. $h^{-1}(4)$ d. $h^{-1}(\{3\mathbb{Z}\})$

(remember that h^{-1} can mean pre-image: it doesn't have to be an inverse function)